PERGAMON

International Journal of Heat and Mass Transfer 45 (2002) 3055–3060

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Employment of non-symmetrical saw-tooth argument transformation method in the elasticity theory for layered composites

G. Starushenko [a,*], N. Krulik [a], S. Tokarzewski [b,1]

[a] Ukrainian State University of Chemical Engineering, Gagarina ave. 8, 49640 Dniepropetrovsk, Ukraine

[b] Institute of Fundamental Technological Research, Swietokrzyska, 21, 00-049 Warsaw, Poland

Received 11 November 2000

Abstract

The analytical method of wave and oscillation theory differential equations periodic solution building on the so-called saw-tooth argument transformation base has been used in the paper series [V.N. Pilipchuk, in: International Congress of Mathematicians, Zurich, 3–11 August 1994, Short Communications, 1994, p. 202; V.N. Pilipchuk, G.A. Starusenko, Izvestiya Nastionaljnoj Akademii Nauk Ukrainy 11 (1997) 25 (in Russian)]. This approach has been developed further, where famous transformation has been generalized on non-symmetrical case. In the present paper the non-symmetrical saw-tooth argument transformation method is suggested to apply to the elasticity theory periodic problems. © 2002 Published by Elsevier Science Ltd.

Keywords: Saw-tooth argument transformation method; Boundary value problem; Layered composite

1. Main mathematical correlation for non-symmetrical saw-tooth argument transformation

Let us denote by $\tau = \tau(x)$ 4-periodic piece-linear (saw-tooth) function defined on the period by the correlation

$$\tau(x) = \begin{cases} k_1 x, & -(1+\theta) \leqslant x \leqslant (1+\theta), \\ k_2(x-2), & (1+\theta) \leqslant x \leqslant (3-\theta), \end{cases} \tag{1}$$

here $k_1 = 1/(1+\theta)$, $k_2 = -1/(1-\theta), -1 \leqslant \theta \leqslant 1$, θ is a parameter which is characterized the "saw" slope (Fig. 1). Value $\theta = 0$ corresponds to symmetrical saw-tooth function [3].

The function $\tau = \tau(x)$ utilization as new argument allows to present any continuous 4a-periodic function $f(x)$ by the following correlation [2]:

$$f(x) = P(\tau) + Q(\tau)\tau', \quad \tau = \tau(x/a), \tag{2}$$

here

$$P(\tau) = 0.5\{(1+\theta)f[(1+\theta)a\tau] + (1-\theta)f[(2-(1-\theta)\tau)a]\},$$
$$Q(\tau) = 0.5\{(1-\theta^2\}\{f[(1+\theta)a\tau] - f[(2-(1-\theta)\tau)a]\}.$$

It is right for any value of x. Identity of correlation (2) is verified by direct substitution on the period.

Observe some properties of non-symmetrical saw-tooth function (1) which are used in the τ-transformation algebra. In so doing we will suggest that the function $f(x)$ in the correlation (2) is 4-periodic one (for 4a-periodic functions all argumentation are remained valid with reference to the substitution $\tau(x) \to a\tau(x/a)$).

1. The function τ' in the expression (2) has discontinuities of the first kind in the points x: $\tau(x) = \pm 1$ (it is not differentiable in classical sense in these points).

Therefore differentiation of the correlation (2)

$$df/dx = P'\tau' + Q'\tau'^2 + Q\tau'' \tag{3}$$

* Corresponding author. Tel.: +380-562-470-813; fax: +380-562-470-813.

E-mail addresses: ugxtu@dicht.dp.ua (G. Starushenko), stokarz@ippt.gov.pl (S. Tokarzewski).

[1] Tel.: +4822-826-12-81; fax: +4822-826-98-15.

0017-9310/02/$ - see front matter © 2002 Published by Elsevier Science Ltd.
PII: S0017-9310(02)00005-4

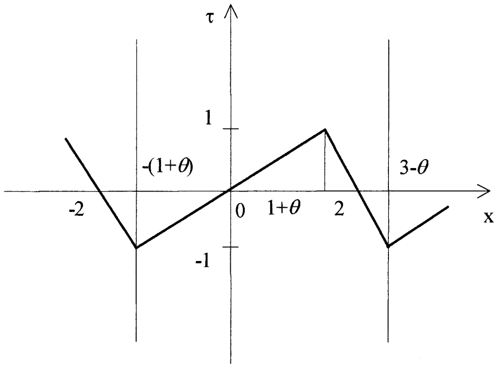

Fig. 1. Graph of saw-tooth function.

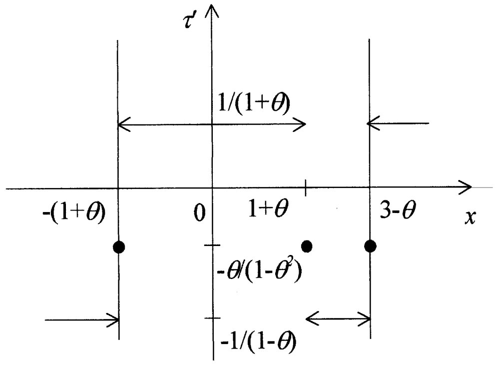

Fig. 2. Graph of saw-tooth function derivative.

leads to appearance of the singular term $Q\tau''$ (specialties of Dirac δ-impulses). They are localized in the points x: $\tau(x) = \pm 1$.

The singular term should be excluded when we consider the function $f(x)$ regularity by the condition adoption

$$Q|_{\tau=\pm 1} = 0. \tag{4}$$

2. Consider function τ'^2 from correlation (3). The expression for the square of derivative τ' can be present in the following form:

$$\tau'^2 = \begin{cases} 1/(1+\theta)^2, & -(1+\theta) < x < (1+\theta), \\ 1/(1-\theta)^2, & (1+\theta) < x < (3-\theta). \end{cases} \tag{5}$$

The correlation (5) can be presented with the help of the function τ' in the unanimous analytical expression form:

$$\tau'^2 = \Delta_1 + \Delta_2\tau', \tag{6}$$

here $\Delta_1 = 1/(1-\theta^2)$, $\Delta_2 = -2\theta/(1-\theta^2)$.

Then the correlation (3) can be presented in the form finally

$$\mathrm{d}f/\mathrm{d}x = \Delta_1 Q' + (P' + \Delta_2 Q')\tau'. \tag{7}$$

3. It is essentially that the expression for τ'^2 (6) (Dirac δ-function which has discontinuity in the points of localization of δ-impulses type function) allows also to determine the correlation $\tau'\tau''$. So when we offer that

$$\tau'\tau'' = 0.5(\tau'^2)' = 0.5(\Delta_1 + \Delta_2\tau')',$$

we will obtain

$$\tau'\tau'' = -\theta/(1-\theta^2)\tau''. \tag{8}$$

(For comparing $\tau'\tau'' = 0$ in the case of symmetrical saw-tooth function. It has been shown yearly [1] and it is obtained from the correlation (8) in particular.)

The correlation (8) allows to determine the value of "saw" in the cone points x: $\tau(x) = \pm 1$. Consequently one obtains that $\tau'(x)$ is determined in the form (Fig. 2):

$$\tau' = \begin{cases} 1/(1+\theta), & -(1+\theta) < x < (1+\theta), \\ -\theta/(1-\theta^2), & x = \pm(1+\theta),\ x = (3-\theta), \\ -1/(1-\theta), & (1+\theta) < x < (3-\theta). \end{cases} \tag{9}$$

2. Elasticity theory differential equations transformation on the periodic solutions set

We will illustrate the application of saw-tooth argument transformation technique to the searching elasticity theory periodic problem solution on the example of two-phase layered composite massif problem. We will consider that the structure is periodic one in the axis Ox direction, for example, and its period is small sufficiently if we compare it with characteristic massif dimension. We will denote composite elastic characteristics – Lame coefficients – as λ_1, μ_1 and λ_2, μ_2 for any of layer accordingly; mass forces as – analogously.

Using definition of saw-tooth function (1) structure the elastic constants and its mass forces can be presented in the form of unanimous analytical expressions which are right for whole layered massif.

$$\lambda^*(\tau) = \lambda(1 + \alpha\tau'), \quad \mu^*(\tau) = \mu(1 + \beta\tau'), \tag{10}$$

here $\tau = \tau(x/a)$; 4a is period of structure.

The correlation (10) can be present in the other form if we present it as

$$\lambda_{1(2)} = \lambda(1 + k_{1(2)}\alpha), \quad \mu_{1(2)} = \mu(1 + k_{1(2)}\beta), \tag{11}$$

here indexes 1,2 indicate on the first or second composite phases accordingly.

Such denomination (11) gives opportunity to change layers elastic characteristics and mass forces if the parameters λ, α, μ, β, and coefficients are modified and to change phases geometric dimensions if the parameter θ is modified.

If we regard the correlation (10) the elasticity theory equilibrium equations in displacement U, V, W can be presented in the form [4]:

$$\begin{aligned}&(\lambda^*+2\mu^*)\partial^2U/\partial x^2+\mu^*(\partial^2U/\partial y^2+\partial^2U/\partial z^2)\\&\quad+(\lambda^*+\mu^*)(\partial^2V/\partial x\partial y+\partial^2W/\partial x\partial z)\\&\quad+(\mathrm{d}\lambda^*/\mathrm{d}x+2\mathrm{d}\mu^*/\mathrm{d}x)\partial U/\partial x\\&\quad+\mathrm{d}\lambda^*/\partial x(\partial V/\partial y+\partial W/\partial z)+X^*=0,\\&(\lambda^*+2\mu^*)\partial^2V/\partial y^2+\mu^*(\partial^2V/\partial x^2+\partial^2V/\partial z^2)\\&\quad+(\lambda^*+\mu^*)(\partial^2U/\partial x\partial y+\partial^2W/\partial y\partial z)\\&\quad+\mathrm{d}\mu^*/\mathrm{d}x(\partial U/\partial y+\partial V/\partial x)+Y^*=0,\\&(\lambda^*+2\mu^*)\partial^2W/\partial z^2+\mu^*(\partial^2W/\partial x^2+\partial^2W/\partial y^2)\\&\quad+(\lambda^*+\mu^*)(\partial^2U/\partial x\partial z+\partial^2V/\partial y\partial z)\\&\quad+\mathrm{d}\mu^*/\mathrm{d}x(\partial U/\partial z+\partial W/\partial x)+Z^*=0.\end{aligned}\tag{12}$$

Taking into account the representation (11) and accounting structure changeable only in the axis Ox direction the periodic problem solution is search in the form

$$U(\tau)=U^{(1)}(\tau)+U^{(2)}(\tau)\tau',\quad \tau=\tau(x/a).\tag{13}$$

Analogous representations take place for compilers V, W if we change $U\to V\to W$. If the correlation (13) is differentiated and correlations (7) and (4) are taken into account we will obtain

$$\begin{aligned}\partial U/\partial x=\{&-k_1k_2\mathrm{d}U^{(2)}/\mathrm{d}\tau+[\mathrm{d}U^{(1)}/\mathrm{d}\tau\\&+(k_1+k_2)\mathrm{d}U^{(2)}/\mathrm{d}\tau]\tau'\}/a,\end{aligned}\tag{14}$$

$$U^{(2)}\mid_{\tau=\pm1}=0.\tag{15}$$

The second derivative is presented by the following correlation:

$$\begin{aligned}\partial^2U/\partial x^2=\{&-k_1k_2\left[\mathrm{d}^2U^{(1)}/\mathrm{d}\tau^2+(k_1+k_2)\mathrm{d}^2U^{(2)}/\mathrm{d}\tau^2\right]\\&+\left[(k_1^2+k_1k_2+k_2^2)\mathrm{d}^2U^{(2)}/\mathrm{d}\tau^2\right.\\&\left.+(k_1+k_2)\mathrm{d}^2U^{(1)}/\mathrm{d}\tau^2\right]\tau'\\&+\left[\mathrm{d}U^{(1)}/\mathrm{d}\tau+(k_1+k_2)\mathrm{d}U^{(2)}/\mathrm{d}\tau\right]\tau''\}/a^2,\end{aligned}\tag{16}$$

where $\partial V/\partial x$, $\partial W/\partial x$, $\partial^2V/\partial x^2$, $\partial^2W/\partial x^2$ are determined analogously.

The expressions (14) and (16) (and analogous ones for functions V, W) and correlations

$$\mathrm{d}\lambda^*/\mathrm{d}x=\lambda\alpha\tau''/a,\quad \mathrm{d}\mu^*/\mathrm{d}x=\mu\beta\tau''/a$$

are substituted into the Eq. (12) and the coefficients under the basis elements 1, τ' are equated. The periodic singular terms are excluded by the condition adoption:

$$\begin{aligned}&\{\mathrm{d}U^{(1)}/\mathrm{d}\tau[(\lambda+2\mu)+(\lambda\alpha+2\mu\beta)(k_1+k_2)]\\&\quad+\mathrm{d}U^{(2)}/\mathrm{d}\tau\left[(\lambda+2\mu)(k_1+k_2)+(\lambda\alpha+2\mu\beta)(k_1^2\right.\\&\quad\left.+k_1k_2+k_2^2)\right]\}\mid_{\tau=\pm1}=0,\quad \{\mathrm{d}V^{(1)}/\mathrm{d}\tau[1+\beta(k_1+k_2)]\\&\quad+\mathrm{d}V^{(2)}/\mathrm{d}\tau[(k_1+k_2)+\beta(k_1^2+k_1k_2+k_2^2)]\}\mid_{\tau=\pm1}=0,\\&\quad(V\to W).\end{aligned}\tag{17}$$

Then the differential equations system for any pairs of functions $U^{(1)}$, $U^{(2)}$, $V^{(1)}$, $V^{(2)}$, $W^{(1)}$, $W^{(2)}$ are obtained.

$$\begin{aligned}&\mathrm{d}^2U^{(1)}/\mathrm{d}\tau^2[(\lambda+2\mu)+(\lambda\alpha+2\mu\beta)(k_1+k_2)]\\&\quad+\mathrm{d}^2U^{(2)}/\mathrm{d}\tau^2[(\lambda+2\mu)(k_1+k_2)\\&\quad+(\lambda\alpha+2\mu\beta)(k_1^2+k_1k_2+k_2^2)]=a^2X^{(1)}/(k_1k_2),\\&\mathrm{d}^2U^{(1)}/\mathrm{d}\tau^2[(\lambda+2\mu)(k_1+k_2)\\&\quad+(\lambda\alpha+2\mu\beta)(k_1^2+k_1k_2+k_2^2)]\\&\quad+\mathrm{d}^2U^{(2)}/\mathrm{d}\tau^2[(\lambda+2\mu)(k_1^2+k_1k_2+k_2^2)\\&\quad+(\lambda\alpha+2\mu\beta)(k_1+k_2)(k_1^2+k_2^2)]=-a^2X^{(2)}.\end{aligned}\tag{18}$$

$$\begin{aligned}&\mathrm{d}^2V^{(1)}/\mathrm{d}\tau^2[1+\beta(k_1+k_2)]+\mathrm{d}^2V^{(2)}/\mathrm{d}\tau^2\\&\quad\left[(k_1+k_2)+\beta(k_1^2+k_1k_2+k_2^2)\right]=a^2Y^{(1)}/(\mu k_1k_2),\\&\mathrm{d}^2V^{(1)}/\mathrm{d}\tau^2\left[(k_1+k_2)+\beta(k_1^2+k_1k_2+k_2^2)\right]\\&\quad+\mathrm{d}^2V^{(2)}/\mathrm{d}\tau^2\left[(k_1^2+k_1k_2+k_2^2)\right.\\&\quad\left.+\beta(k_1+k_2)(k_1^2+k_2^2)\right]=-a^2Y^{(2)}/\mu,\quad (V\to W).\end{aligned}\tag{19}$$

By obvious transformations any of these systems splits on the differential equations for functions $U^{(1)}$, $U^{(2)}(U\to V\to W)$. Thus, if we regard denomination (11) the elasticity theory periodic problem for two-phase layered composite is reduced to the following boundary value problems:

$$\begin{aligned}&\mathrm{d}^2U^{(1)}/\mathrm{d}\tau^2=T_{U1}^{(1)}X_1+T_{U1}^{(2)}X_2,\\&\mathrm{d}^2U^{(2)}/\mathrm{d}\tau^2=T_{U2}^{(1)}X_1+T_{U2}^{(2)}X_2,\\&U^{(2)}\mid_{\tau=\pm1}=0,\\&\{\mathrm{d}U^{(1)}/\mathrm{d}\tau T_{U1}+\mathrm{d}U^{(2)}/\mathrm{d}\tau T_{U2}\}\mid_{\tau=\pm1}=0,\end{aligned}\tag{20}$$

where

$$\begin{aligned}&T_{U1}^{(1)}=a^2k_2/\left[(k_1-k_2)k_1^2(\lambda_1+2\mu_1)\right],\\&T_{U1}^{(2)}=T_{U1}^{(1)}(1\leftrightarrow2),\\&T_{U2}^{(1)}=-T_{U1}^{(1)}/k_2,\quad T_{U2}^{(2)}=T_{U2}^{(1)}(1\leftrightarrow2),\\&T_{U1}=k_1(\lambda_1+2\mu_1)-k_2(\lambda_2+2\mu_2),\\&T_{U2}=k_1^2(\lambda_1+2\mu_1)-k_2^2(\lambda_2+2\mu_2).\end{aligned}$$

$$\begin{aligned}&\mathrm{d}^2V^{(1)}/\mathrm{d}\tau^2=T_{V1}^{(1)}Y_1+T_{V1}^{(2)}Y_2,\\&\mathrm{d}^2V^{(2)}/\mathrm{d}\tau^2=T_{V2}^{(1)}Y_1+T_{V2}^{(2)}Y_2,\\&V^{(2)}\mid_{\tau=\pm1}=0,\\&\{\mathrm{d}V^{(1)}/\mathrm{d}\tau T_{V1}+\mathrm{d}V^{(2)}/\mathrm{d}\tau T_{V2}\}\mid_{\tau=\pm1}=0,\end{aligned}\tag{21}$$

where

$$
\begin{aligned}
&T_{V1}^{(1)} = a^2k_2/\left[(k_1-k_2)k_1^2\mu_1\right],\\
&T_{V1}^{(2)} = T_{V1}^{(1)}(1\leftrightarrow 2),\quad T_{V2}^{(1)} = -T_{V1}^{(1)}/k_2,\\
&T_{V2}^{(2)} = T_{V2}^{(1)}(1\leftrightarrow 2),\\
&T_{V1} = \mu_1k_1-\mu_2k_2,\quad T_{V2} = \mu_1k_1^2-\mu_2k_2^2,\\
&(V\to W).
\end{aligned}
$$

We will consider the limit cases of the boundary value problems (20) and (21). They depend on the rigid and geometrical composite phases characteristics. We will consider that characteristics of the first phase are fixed ones and characteristics of the second phase (inclusion) are variable ones.

1. Equilateral phases: $k_1 = 1$, $k_2 = -1$. Then

$$
\begin{aligned}
&T_{U1}^{(1)} = -a^2/[2(\lambda_1+2\mu_1)],\\
&T_{U1}^{(2)} = T_{U1}^{(1)}(1\leftrightarrow 2),\quad T_{U2}^{(1)} = T_{U1}^{(1)},\\
&T_{U2}^{(2)} = -T^{(1)}r_{U2}(1\leftrightarrow 2),\\
&T_{U1} = (\lambda_1+\lambda_2)+2(\mu_1+\mu_2),\\
&T_{U2} = (\lambda_1-\lambda_2)+2(\mu_1-\mu_2),\\
&T_{V1}^{(1)} = -0.5a^2/\mu_1,\\
&T_{V1}^{(2)} = T_{V1}^{(1)}(1\leftrightarrow 2),\quad T_{V2}^{(1)} = T_{V1}^{(1)},\\
&T_{V2}^{(2)} = -T_{V2}^{(1)}(1\leftrightarrow 2),\quad T_{V1} = \mu_1+\mu_2,\\
&T_{V2} = \mu_1-\mu_2.
\end{aligned}
\tag{22}
$$

2. Homogeneous structure with the periodic mass forces: $\lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu$. Then

$$
\begin{aligned}
&T_{U1}^{(1)} = a^2k_2/\left[(k_1-k_2)k_1^2(\lambda+2\mu)\right],\\
&T_{U1}^{(2)} = T_{U1}^{(1)}(1\leftrightarrow 2),\\
&T_{U2}^{(1)} = -T_{U1}^{(1)}/k_2,\quad T_{U2}^{(2)} = T_{U2}^{(1)}(1\leftrightarrow 2),\\
&T_{U1} = (k_1-k_2)(\lambda+2\mu),\\
&T_{U2} = T_{U1}(k_1+k_2),\\
&T_{V1}^{(1)} = a^2k_2/[(k_1-k_2)k_1^2\mu],\\
&T_{V1}^{(2)} = T_{V1}^{(1)}(1\leftrightarrow 2),\\
&T_{V2}^{(1)} = -T_{V1}^{(1)}/k_2,\quad T_{V2}^{(2)} = T_{V2}^{(1)}(12),\\
&T_{V1} = \mu(k_1-k_2),\quad T_{V2} = T_{V1}(k_1+k_2).
\end{aligned}
\tag{23}
$$

3. Absolute rigid inclusions: $\lambda_2/\lambda_1\to\infty$, $\mu_2/\mu_1\to\infty$. Then

$$
\begin{aligned}
&T_{U1}^{(1)} = a^2k_2/\left[(k_1-k_2)k_1^2(\lambda_1+2\mu_1)\right],\\
&T_{U1}^{(2)} = 0,\quad T_{U2}^{(1)} = -T_{U1}^{(1)}/k_2,\\
&T_{U2}^{(2)} = 0,\quad T_{U1} = 1,\quad T_{U2} = k_2,\\
&T_{V1}^{(1)} = a^2k_2/[(k_1-k_2)k_1^2\mu_1],\\
&T_{V1}^{(2)} = 0,\quad T_{V2}^{(1)} = -T_{V1}^{(1)}/k_2,\\
&T_{V2}^{(2)} = 0,\quad T_{V1} = 1,\quad T_{V2} = k_2.
\end{aligned}
\tag{24}
$$

4. Soft inclusions: $\lambda_2/\lambda_1 \ll 1, \mu_2/\mu_1 \ll 1$. Then

$$
\begin{aligned}
&T_{U1}^{(1)} = 0,\\
&T_{U1}^{(2)} = -a^2k_1/[(k_1-k_2)k_2^2(\lambda_2+2\mu_2)],\\
&T_{U2}^{(1)} = 0,\quad T_{U2}^{(2)} = -T_{U1}^{(2)}/k_1,\quad T_{U1} = 1,\\
&T_{U2} = k_1,\quad T_{V1}^{(1)} = 0,\\
&T_{V1}^{(2)} = -a^2k_1/[(k_1-k_2)k_2^2\mu_2],\\
&T_{V2}^{(1)} = 0,\quad T_{V2}^{(2)} = -T_{V1}^{(2)}/k_1,\quad T_{V1} = 1,\\
&T_{V2} = k_2.
\end{aligned}
\tag{25}
$$

5. Soft inclusions which are free ones from mass forces (in cavity limit): $\lambda_2/\lambda_1\to 0$, $\mu_2/\mu_1\to 0$, $X_2 = Y_2 = Z_2 = 0$. Then

$$
\begin{aligned}
&T_{U1}^{(1)} = a^2k_2/\left[(k_1-k_2)k_1^2(\lambda_1+2\mu_1)\right],\\
&T_{U1}^{(2)} = 0,\quad T_{U2}^{(1)} = -T_{U1}^{(1)}/k_2,\\
&T_{U1}^{(2)} = 0,\quad T_{U1} = 1,\quad T_{U2} = k_1,\\
&T_{V1}^{(1)} = a^2k_2/\left[(k_1-k_2)k_1^2\mu_1\right],\\
&T_{V1}^{(2)} = 0,\quad T_{V2}^{(1)} = -T_{V1}^{(1)}/k_2,\\
&T_{V2}^{(2)} = 0,\quad T_{V1} = 1,\quad T_{V2} = k_1.
\end{aligned}
\tag{26}
$$

6. Non-deformed thin inclusions (plates): $\lambda_2/\lambda_1\to\infty$, $\mu_2/\mu_1\to\infty$, $k_1 = 0.5, k_2\to-\infty$. Then

$$
\begin{aligned}
&T_{U1}^{(1)} = -4a^2/(\lambda_1+2\mu_1),\quad T_{U1}^{(2)} = 0,\\
&T_{U2}^{(1)} = -T_{U1}^{(1)}/k_2,\quad T_{U2}^{(2)} = 0,\quad T_{U1} = 1,\\
&T_{U2} = k_2,\quad T_{V1}^{(1)} = -4a^2/\mu_1,\quad T_{V1}^{(2)} = 0,\\
&T_{V2}^{(1)} = -T_{V1}^{(1)}/k_2,\quad T_{V2}^{(2)} = 0,\\
&T_{V1} = 1,\quad T_{V2} = k_2.
\end{aligned}
\tag{27}
$$

7. Soft thin plates which are free ones from mass forces: $\lambda_2/\lambda_1\to 0$, $\mu_2/\mu_1\to 0$, $k_1 = 0.5, k_2\to-\infty$, $X_2 = Y_2 = Z_2 = 0$.

In this case, it is possible the following asymptotic representation in dependence on geometrical and rigid parameters:

$$
\begin{aligned}
&T_{U1}^{(1)} = -4a^2/(\lambda_1+2\mu_1),\\
&T_{U1}^{(2)} = 0,\quad T_{U2}^{(1)} = -T_{U1}^{(1)}/k_2,\quad T_{U2}^{(2)} = 0,\\
&T_{V1}^{(1)} = -4a^2/\mu_1,\quad T_{V1}^{(2)} = 0,\\
&T_{V2}^{(1)} = -T_{V1}^{(1)}/k_2,\quad T_{V2}^{(2)} = 0.
\end{aligned}
\tag{28}
$$

(a) $k_2\lambda_2/\lambda_1 \ll 1/k_2(\lambda\to\mu)$

$$T_{U1} = T_{V1} = 1,\quad T_{U2} = T_{V2} = 0.5. \tag{28.1}$$

(b) $k_2\lambda_2/\lambda_1 \sim 1/k_2(\lambda\to\mu)$

$$
\begin{aligned}
&T_{U1} = 0.5(\lambda_1+2\mu_1),\\
&T_{U2} = 0.25(\lambda_1+2\mu_1)-k_2^2(\lambda_2+2\mu_2),\\
&T_{Vi} = T_{Ui}(\lambda_{1(2)}+2\mu_{1(2)}\to\mu_{1(2)}),\quad i = 1,2.
\end{aligned}
\tag{28.2}
$$

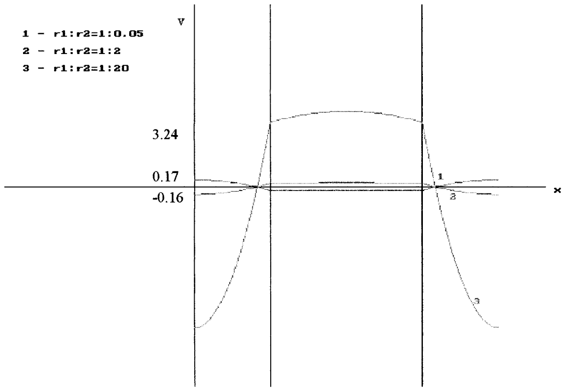

Fig. 3. Graph of transference $V(\tau)$ under $\theta = 0, \mu 1 = 1, \mu 2 = 20$.

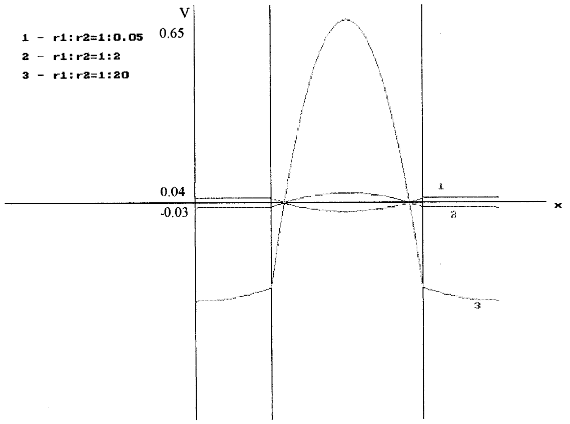

Fig. 4. Graph of transference under $\theta = 0$, $\mu 1 = 1$, $\mu 2 = 0.05$.

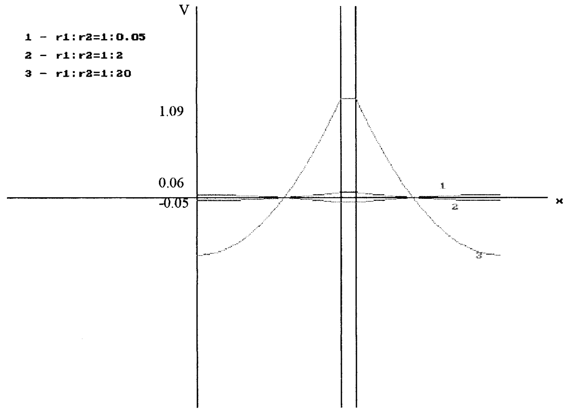

Fig. 5. Graph of transference under $\theta = 0.9$, $\mu 1 = 1$, $\mu 2 = 20$.

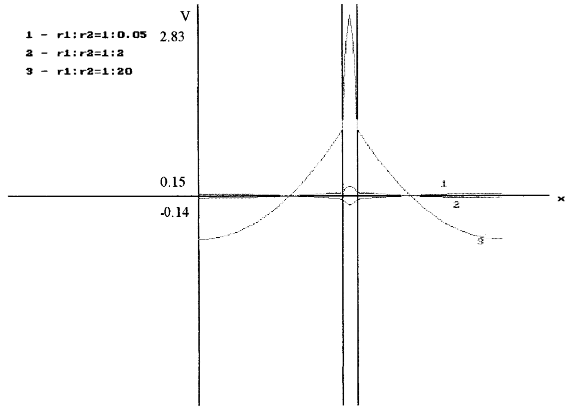

Fig. 6. Graph of transference under $\theta = 0.9$, $\mu 1 = 1$, $\mu 2 = 0.05$.

(c) $1/k_2 \ll k_2\lambda_2/\lambda_1 \ll 1(\lambda \to \mu)$

$$T_{U1} = 0.5(\lambda_1 + 2\mu_1), \quad T_{U2} = -k_2^2(\lambda_2 + 2\mu_2), \\ T_{Vi} = T_{Ui}(\lambda_{1(2)} + 2\mu_{1(2)} \to \mu_{1(2)}), \quad i = 1,2. \tag{28.3}$$

(d) $k_2\lambda_2/\lambda_1 \sim 1(\lambda \to \mu)$

$$T_{U1} = 0.5(\lambda_1 + 2\mu_1) - k_2(\lambda_2 + 2\mu_2), \\ T_{U2} = -k_2^2(\lambda_2 + 2\mu_2), \\ T_{Vi} = T_{Ui}(\lambda_{1(2)} + 2\mu_{1(2)} \to \mu_{1(2)}), \quad i = 1,2. \tag{28.4}$$

(e) $k_2/\lambda_2/\lambda_1 \gg 1(\lambda \to \mu)$

$$T_{U1} = T_{V1} = 1, \quad T_{U2} = T_{V2} = k_2. \tag{28.5}$$

It is worth to note that the boundary value problems (20) and (21) (or analogous ones for them in particular cases (22)–(27), (28), (28.1)–(28.5) are compatible ones and its solutions are determined up to any constant if mass forces are self-balanced ones on the period or if the following conditions

$$\int_{-1}^{1} X^{(1)}(\tau)\,d\tau = 0\big(X^{(1)} \to Y^{(1)} \to Z^{(1)}\big) \tag{29}$$

are satisfied.

Using condition of "slaw" (average) component exclusion from the periodic solution which is determined by the correlation

$$\int_{-1}^{1} U^{(1)}(\tau)\,d\tau = 0(U \to V \to W) \tag{30}$$

we will obtain only periodic problem solution which is defined by the only way finally.

The plain problem solution for two phases 4-periodic structure under mass force effects in the axis Oy direction has been obtained as illustrative example

$$Y_{1(2)} = r_{1(2)}.$$

In this case components of the function $U = W \equiv O$. The problem is reduced to functions $V^{(1)}(\tau)$, $V^{(2)}(\tau)$ determination from the boundary value problem (21) (if $a = l$). From the correlation (29) we will obtain:

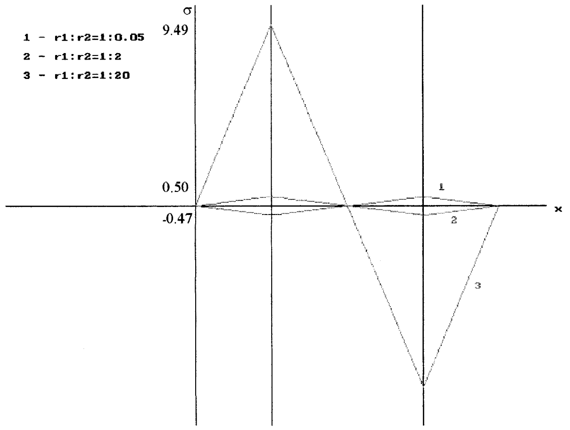

Fig. 7. Graph of efforts σ_{xy} under $\theta = 0$, $\mu 1 = 1$, $\mu 2 = 20$.

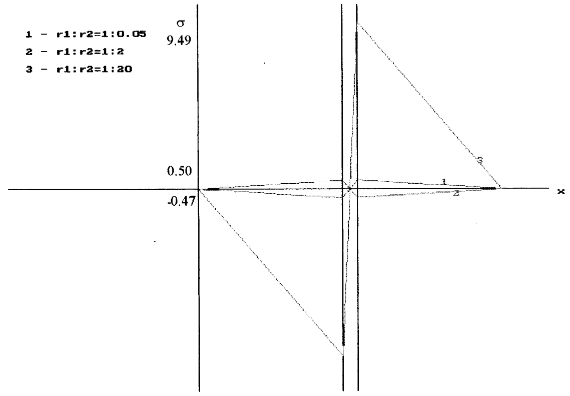

Fig. 8. Graph of efforts σ_{xy} under $\theta = 0.9$, $\mu 1 = 1$, $\mu 2 = 20$.

$$Y_{1(2)} = r_{1(2)} - 1/2(r_1/k_1 - r_2/k_2) = 1/(2k_{2(1)})(r_2 - r_1). \tag{31}$$

This boundary value problem solution was obtained in the form:

$$\begin{aligned} V^{(1)}(\tau) &= \left((r_1 - r_2)\left(\mu_1 k_1^2 - \mu_2 k_2^2\right)/\left(24k_1^3k_2^3\mu_1\mu_2\right)\right) \times (1 - 3\tau^2), \\ V^{(2)}(\tau) &= \left((r_2 - r_1)(\mu_1 k_1 - \mu_2 k_2)/\left(8k_1^3k_2^3\mu_1\mu_2\right)\right) \times (1 - \tau^2). \end{aligned} \tag{32}$$

The displacements $V(\tau)$ graphs in dependence on the different correlations of geometric and rigid characteristic of composites phases you can see in Figs. 3–6 (θ defines geometric characteristic and μ_1, μ_2 define rigid characteristics; r_1, r_2 define mass forces).

Efforts σ_{xy} are determined from the correlation:

$$\begin{aligned} \sigma_{xy} = {} & -(k_1k_2/(k_1 - k_2))[(\mu_1 - \mu_2)\partial V^{(1)}/\partial\tau \\ & + (\mu_1 k_1 - \mu_2 k_2)\partial V^{(2)}/\partial\tau] + (1/(k_1 - k_2)) \\ & \times [(\mu_1 k_1 - \mu_2 k_2)\partial V^{(1)}/\partial\tau + (\mu_1 k_1^2 - \mu_2 k_2^2)\partial V^{(2)}/\partial\tau]\tau'. \end{aligned} \tag{33}$$

In this case we will obtain:

$$\sigma_{xy} = -(r_1 - r_2)(k_1 - k_2)/(4k_1^2k_2^2).$$

Its graphs you can see on the Figs. 7 and 8.

References

[1] V.N. Pilipchuk, On some form of periodic solutions representation (non-smooth transformation of arguments, the corresponding algebraic structure, and applications), in: International Congress of Mathematicians, Zurich, 3-11 August 1994. Short Communications, 1994, p. 202.

[2] V.N. Pilipchuk, G.A. Starusenko, About representation of periodic solutions of differential equations by sidelong corner saw-tooth argument transformation, Izvestiya Nastionaljnoj Akademii Nauk Ukrainy (11) (1997) 25–28 (in Russian).

[3] V.N. Pilipchuk, To calculation of mechanical systems with impulses stimulus, Prikladnaya matematika i mexanika 60 (2) (1996) 223–232 (in Russian).

[4] S.P. Timoshenko, in: Course of Elasticity Theory, Naukova dumka, Kiev, 1972, p. 508 (in Russian).